Combining relations (2) and (5) we obtain

$$k = q/4. \tag{6}$$

Equating the values of k given by relations (5) and (6) we find that p and q are connected by the linear equation

$$2p - q + 2 = 0. \tag{7}$$

Substituting k from relation (6) in relation (3) and then employing equation (7) we find

$$r = -3q^2/16 = -3(p+1)^2/4. \tag{8}$$

Incidentally, the last result shows that r must be negative in order that p and q may be real.

Relations (7) and (8) enable the given quartic to be expressed in terms of (say) p alone, namely

$$x^4 + px^2 + 2(p+1)x - 3(p+1)^2/4 = 0,$$

or

$$[x^2 + (p+1)/2]^2 = [x - (p+1)]^2.$$

Consequently, the roots of the quartic are

$$x = (1 \pm \sqrt{-6p-5})/2,$$

$$x = (-1 \pm \sqrt{2p+3})/2.$$

For $-3/2 \leqq p \leqq -5/6$ all four roots will be real. For values of p outside of this range two roots will be real and two complex.

The reduction of the general quartic to the difference of two squares is given in Burnside and Panton's *Theory of Equations*, vol. I, p. 129, Art. 63 (4th or 7th editions.)

2783 [1919, 312]. Proposed by C. C. BRAMBLE, U. S. Naval Academy.

Two players, A and B, take turns throwing a single dice, A leading. The one first making a score of three aces is to be the winner. Find the probability that A will win.

SOLUTION BY H. P. MANNING, Brown University.

The probability of throwing three aces in the first three throws is $1/6^3$; in general, the probability of throwing exactly two aces in the first n throws, and an ace in the $(n+1)$th throw, is $\frac{n(n-1)}{2} \cdot \frac{5^{n-2}}{6^{n+1}}$. That is, the probabilities of the different ways of throwing the first three aces are for A or B

$$\frac{1}{6^3},\quad 3\frac{5}{6^4},\quad 6\frac{5^2}{6^5},\quad \cdots.$$

Call these numbers

$$a \quad b \quad c \quad \cdots.$$

Their sum should equal 1.

The probability that A wins is

$$a + (1-a)b + (1-a-b)c + \cdots,$$

the factor in the parenthesis in any term indicating the probability that B has not yet thrown three aces. This expression may be written

$$a + b + c + \cdots$$
$$-(ab + ac + bc + \cdots).$$

Similarly the probability that B wins is

$$(1-a)a + (1-a-b)b + \cdots,$$

which may be written

$$a + b + c + \cdots$$
$$-(a^2 + ab + b^2 + ac + bc + c^2 + \cdots).$$

Put

$$M = a + b + c + \cdots$$
$$N = a^2 + b^2 + c^2 + \cdots$$
$$P = ab + ac + bc + \cdots,$$

$$M = \frac{1}{2\cdot 6^3}\left[1\cdot 2 + 2\cdot 3\cdot \frac{5}{6} + 3\cdot 4\left(\frac{5}{6}\right)^2 + \cdots\right],$$

$$N = \frac{1}{4\cdot 36^3}\left[(1\cdot 2)^2 + (2\cdot 3)^2\frac{25}{36} + (3\cdot 4)^2\left(\frac{25}{36}\right)^2 + \cdots\right].$$

Let

$$f(x) = 1 + x + x^2 + \cdots = \frac{1}{1-x} \quad (|x| < 1);$$

then

$$f''(x) = 1\cdot 2 + 2\cdot 3x + 3\cdot 4x^2 + \cdots = \frac{2}{(1-x)^3}.$$

Hence putting $x = \frac{5}{6}$ we have $M = \frac{1}{2\cdot 6^3}\cdot\frac{2}{(1/6)^3} = 1.$

$$x^2 f''(x) = 1\cdot 2x^2 + 2\cdot 3x^3 + 3\cdot 4x^4 + \cdots.$$

Differentiating twice,

$$2f''(x) + 4xf'''(x) + x^2 f^{\text{iv}}(x) = (1\cdot 2)^2 + (2\cdot 3)^2 x + (3\cdot 4)^2 x^2 + \cdots.$$

This equals

$$2\frac{2}{(1-x)^3} + 4x\frac{6}{(1-x)^4} + x^2\frac{24}{(1-x)^5} = \frac{4}{(1-x)^5}(1 + 4x + x^2).$$

Hence putting $x = \frac{25}{36}$ we have

$$N = \frac{1}{4\cdot 36^3}\cdot\frac{4\cdot 36^5}{11^5}\left(1 + 4\frac{25}{36} + \frac{25^2}{36^2}\right) = \frac{36^2 + 4\cdot 25\cdot 36 + 25^2}{11^5} = \frac{5521}{11^5}.$$

$$M^2 = N + 2P = 1;$$

hence

$$P = \frac{1}{2} - \frac{5521}{2\cdot 11^5}.$$

Now the probability that A wins is $1 - P$ and the probability that B wins is $1 - N - P$. and so we may say, the probability that A wins is $N + P$ and the probability that B wins is P; These probabilities are

$$\text{for } A \quad \frac{1}{2} + \frac{5521}{2\cdot 11^5},$$

$$\text{“} \quad B \quad \frac{1}{2} - \frac{5521}{2\cdot 11^5}.$$

All of these series are absolutely convergent power series, the parentheses may be removed and the terms rearranged as shown in the expressions for the two probabilities, and M may be squared to give the relation $M^2 = N + 2P$.